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Estimates for n-widths of the Hardy-type operators (Addendum to "Improved estimates for the approximation numbers of the Hardy-type operators")

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Abstract

Consider the Hardy-type operator $T: L^p(a,b) \to L^p(a,b)$, $-\infty \leqslant a < b \leqslant \infty$, which is defined by

$$(Tf)(x) = v(x)\int_a^x u(t)f(t)\,dt.$$

It is shown that

$$\rho_n(T) = \frac{1}{n}\alpha_p \int_a^b u(x)v(x) + O(n^{-2}),$$

where $\rho_n(T)$ stands for any of the following: the Kolmogorov n-width, the Gel'fand n-width, the Bernstein n-width or the nth approximation number of T.

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The aim of this note is to improve and extend results from [1]. We obtain sharper estimates for the approximation numbers $a_n(T)$ and also for the n-widths of T, where T stands for the weighted Hardy-type integral operator $T : L^p(a,b) \to L^p(a,b)$, $-\infty \leqslant a < b \leqslant \infty$, which is defined by

$$(Tf)(x) = v(x)\int_a^x u(t)f(t)\,dt \quad \text{where } u, v > 0. \tag{1}$$

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We keep the notation from [1]. We recall that:

Definition 1. Let $T : L^p(I) \to L^p(I)$ be a bounded operator and $n \in \mathbf{N}$.

(i) The Kolmogorov n-width $d_n(T)$ of T is defined by

$$d_n(T) = d_n(T(L^p(I)), L^p(I)) = \inf_{X_n} \sup_{\|x\|_{L^p(I)} \leqslant 1} \inf_{y \in X_n} \|Tx - y\|_{L^p(I)},$$

where the infimum is taken over all n-dimensional subspaces X_n of $L^p(I)$.

(ii) The Gel'fand n-width $d^n(T)$ of T is defined by

$$d^n(T) = d^n(T(L^p(I)), L^p(I)) = \inf_{L^n} \sup_{\|x\|_{L^p(I)} \leqslant 1, x \in L^n} \|Tx\|_{L^p(I)},$$

where the infimum is taken over all subspaces of codimension n L_n of $L^p(I)$.

(iii) The Bernstein n-width $b_n(T)$ of T is defined by

$$b_n(T) = b_n(T(L^p(I)), L^p(I)) = \sup_{X_{n+1}} \inf_{Tx \in X_{n+1}, Tx \neq 0} \|Tx\|_{L^p(I)} / \|x\|_{L^p(I)},$$

where X_{n+1} is any subspace of span $\{Tx : x \in X\}$ of dimension $\geqslant n+1$.

Ref. [2] gives us the following relation between the approximation numbers and n-widths:

$$a_{n+1}(T) \geqslant d_n(T), \quad d^n(T) \geqslant b_n(T). \tag{2}$$

We now prove the first essential lemma of this note.

Lemma 2. *Let* $\varepsilon > 0$, $1 < p < \infty$ *and let* $I = (a, b)$. *If* $N := N(\varepsilon)$; *then* $b_{N-2}(T) \geqslant \varepsilon$.

Proof. From the definition of $N(\varepsilon)$ we have that for $i = 1, \ldots, N-1$, $A(I_i) = \varepsilon$. Let $\lambda \in (0, 1)$. Then from the definition of $A(I_i)$ we have, for each $i = 1, \ldots, N-1$, a function $\phi_i \in L^p(I)$, where $\|\phi_i\|_{p,I} = 1$, with support in I_i such that

$$\inf_{\alpha \in \mathbf{R}} \|T\phi_i - \alpha v\|_{p,I_i} > \lambda A(I_i) \geqslant \lambda\varepsilon.$$

Let $X_{N-1} = \text{span}\{T\phi; \phi = \sum_{i=1}^{N-1} \lambda_i \phi_i, \lambda_i \in \mathbf{R}\}$. Then we can see that rank $X_{N-1} \geqslant N-1$. Taking $0 \neq T\phi \in X_{N-1}$, then $0 \neq \phi = \sum_{i=1}^{N-1} \lambda_i \phi_i$ with $\lambda_i \neq 0$ for some i.

$$\begin{aligned}
\|T\phi\|_{p,I}^p &\geqslant \sum_{i=1}^{N-1} \|(T\phi)\chi_{I_i}\|_{p,I}^p \\
&= \sum_{i=1}^{N-1} \left\| \chi_{I_i}(x) v(x) \left(\int_{a_i}^{x} \lambda_i \phi_i(t) \chi_{I_i}(t)\, dt + \int_a^{a_i} \phi(t) u(t)\, dt \right) \right\|_{p,I}^p \\
&= \sum_{i=1}^{N-1} \left\| \left(T\phi_i(x) + v(x) \frac{\eta_i}{\lambda_i} \right) \lambda_i \right\|_{p,I_i}^p, \quad \text{where } \eta_i := \int_a^{a_i} \phi(t) u(t)\, dt \\
&\geqslant \sum_{i=1}^{N-1} \inf_{\alpha \in \mathbf{R}} \|T\phi_i(x) - v(x)\alpha\|_{p,I_i}^p |\lambda_i|^p
\end{aligned}$$

$$\geqslant (\lambda\varepsilon)^p \sum_{i=1}^{N-1} \|\phi_i\|_{p,I_i}^p |\lambda_i|^p \geqslant (\lambda\varepsilon)^p \|\phi\|_{p,I}^p,$$

and the lemma follows. □

From the previous lemma and [1, Lemma 2.6] the next inequality follows.

$$a_{N+1}(T) \leqslant \varepsilon \leqslant b_{N-2}(T) \quad \text{where } N := N(\varepsilon). \tag{3}$$

This with (2) give us

$$a_{N+1}(T) \leqslant \varepsilon \leqslant a_{N-1}(T) \quad \text{and} \quad \rho_N(T) \leqslant \varepsilon \leqslant \rho_{N-2}(T), \tag{4}$$

where $\rho_N(T)$ stands for any of the following $d_N(T)$, $d^N(T)$ or $b_N(T)$.
By [1, Lemma 2.7], (3) and (4) we have by techniques from [1]:

Theorem 3. *Given $v \in L^p(a,b)$, $u \in L^{p'}(a,b)$ the operator T defined in (1) satisfies*

$$\lim_{n\to\infty} n\rho_n(T) = \alpha_p \int_a^b |u(t)v(t)|\,dt,$$

where $\alpha_p = A((0,1),1,1)$ and $\rho_n(T)$ stands for any of the following: $b_n(T)$, $d_n(T)$, $d^n(T)$ or $a_n(T)$.

Inequalities (3), (4) and Theorem 3 with techniques used in the proof of [1, Theorem 4.1] give us the following theorem which is an extension of the main theorem from [1].

Theorem 4. *Let $-\infty \leqslant a < b \leqslant \infty$ and $I = (a,b)$. Let $u \in L^{p'}(I)$, $v \in L^p(I)$ and suppose that $u' \in L^{p'/(p'+1)}(a,b) \cap C([a,b])$, $v' \in L^{p/(p+1)}(a,b) \cap C([a,b])$. Then*

$$\limsup_{n\to\infty} \left| \alpha_p \int_a^b |u(t)v(t)|\,dt - \rho_n(T)n \right| n^{1/2}$$
$$\leqslant c(p,p') \left(\|u'\|_{p'/(p'+1),I} + \|v'\|_{p/(p+1),I} \right) \left(\|u\|_{p',I} + \|v\|_{p,(a,b)} \right) + 3\alpha_p \|uv\|_{1,I},$$

where $\alpha_p = A((0,1),1,1)$, $c(p,p')$ is a constant depending only on p and p' and $\rho_n(T)$ stands for any of the following: $d_n(T)$, $d^n(T)$, $b_n(T)$ or $a_n(T)$.

In the next lemma we improve [1, Theorem 4.1].

Lemma 5. *Let $1 < p < \infty$, $I = (a,b)$, $u \in L^{p'}(I)$, $v \in L^p(I)$ and $v'/v, u'/u \in L^1(I) \cap C[a,b]$. Then*

$$\lim_{\varepsilon\to 0_+} \left| N(\varepsilon) \left[\varepsilon N(\varepsilon) - \alpha_p \int_I u(x)v(x)\,dx \right] \right| \leqslant \int_I u(x)v(x)\,dx \left[\int_I \left| \frac{v'(x)}{v(x)} \right| dx \right.$$
$$+ \int_I \left| \frac{u'(x)}{u(x)} \right| dx + \alpha_p + \int_I \left| \frac{u'(x)}{u(x)} \right| dx$$
$$\left. \times \int_I \left| \frac{v'(x)}{v(x)} \right| dx \right].$$

Proof. Take $\|T\| > \varepsilon > 0$ and $N := N(\varepsilon)$. Then we have the following partition: $I = \bigcup_{i=1}^{N} I_i$, $A(I_i) = \varepsilon$ for $i = \{1, \dots, N-1\}$ and $A(I_N) < \varepsilon$. Define the following step functions:

$$u^{+,\varepsilon}(x) = \sum_{i=1}^{N} u_i^{+,\varepsilon}\chi_{I_i}(x), \quad v^{+,\varepsilon}(x) = \sum_{i=1}^{N} v_i^{+,\varepsilon}\chi_{I_i}(x),$$

$$u^{-,\varepsilon}(x) = \sum_{i=1}^{N} u_i^{-,\varepsilon}\chi_{I_i}(x), \quad v^{-,\varepsilon}(x) = \sum_{i=1}^{N} v_i^{-,\varepsilon}\chi_{I_i}(x),$$

where $u_i^{+,\varepsilon} = \sup_{x\in I_i} |u(x)|$, $u_i^{-,\varepsilon} = \inf_{x\in I_i} |u(x)|$, $v_i^{+,\varepsilon} = \sup_{x\in I_i} |v(x)|$, $v_i^{-,\varepsilon} = \inf_{x\in I_i} |v(x)|$.

Then we have

$$v_i^{+,\varepsilon} - v_i^{-,\varepsilon} \leqslant |I_i| \max_{x\in I_i} |v'(x)| \quad \text{and} \quad u_i^{+,\varepsilon} - u_i^{-,\varepsilon} \leqslant |I_i| \max_{x\in I_i} |u'(x)| \tag{5}$$

and from [1, Lemma 3.1] follows:

$$\alpha_p u_i^{-,\varepsilon} v_i^{-,\varepsilon} |I_i| \leqslant A(I_i) = \varepsilon \leqslant \alpha_p u_i^{+,\varepsilon} v_i^{+,\varepsilon} |I_i|, \tag{6}$$

and we can also see that

$$\int_I u^{-,\varepsilon}(x)v^{-,\varepsilon}(x)\,dx \leqslant \int_I u(x)v(x)\,dx \leqslant \int_I u^{+,\varepsilon}(x)v^{+,\varepsilon}(x)\,dx.$$

Let us estimate the upper bound for the following quantity:

$$\begin{aligned}
K(\varepsilon) &:= \int_I (u^{+,\varepsilon}(x)v^{+,\varepsilon}(x) - u^{-,\varepsilon}(x)v^{-,\varepsilon}(x))\,dx \\
&= \sum_{i=1}^{N} |I_i|(u_i^{+,\varepsilon}v_i^{+,\varepsilon} - u_i^{-,\varepsilon}v_i^{-,\varepsilon}) \\
&= \sum_{i=1}^{N} |I_i|(u_i^{+,\varepsilon}v_i^{+,\varepsilon} - u_i^{+,\varepsilon}v_i^{-,\varepsilon} + u_i^{+,\varepsilon}v_i^{-,\varepsilon} - u_i^{-,\varepsilon}v_i^{-,\varepsilon}) \quad \text{(use (5))} \\
&\leqslant \sum_{i=1}^{N} |I_i| \left[u_i^{+,\varepsilon}|I_i| \max_{x\in I_i} |v'(x)| + v_i^{-,\varepsilon}|I_i| \max_{x\in I_i} |u'(x)| \right] \quad \text{(use (6))} \\
&\leqslant \frac{\varepsilon}{\alpha_p} \sum_{i=1}^{N} \left[\frac{u_i^{+,\varepsilon}}{u_i^{-,\varepsilon}} |I_i| \frac{\max_{x\in I_i} |v'(x)|}{v_i^{-,\varepsilon}} + \frac{v_i^{-,\varepsilon}}{v_i^{-,\varepsilon}} |I_i| \frac{\max_{x\in I_i} |u'(x)|}{u_i^{-,\varepsilon}} \right] \quad \text{(use (5))} \\
&\leqslant \frac{\varepsilon}{\alpha_p} \sum_{i=1}^{N} \left[\frac{u_i^{-,\varepsilon} + |I_i| \max_{x\in I_i} |u'(x)|}{u_i^{-,\varepsilon}} |I_i| \frac{\max_{x\in I_i} |v'(x)|}{v_i^{-,\varepsilon}} \right] \\
&\quad + \frac{\varepsilon}{\alpha_p} \sum_{i=1}^{N} \left[|I_i| \frac{\max_{x\in I_i} |u'(x)|}{u_i^{-,\varepsilon}} \right] \\
&\leqslant \frac{\varepsilon}{\alpha_p} \sum_{i=1}^{N} \left[1 + |I_i| \frac{\max_{x\in I_i} |u'(x)|}{u_i^{-,\varepsilon}} \right] |I_i| \frac{\max_{x\in I_i} |v'(x)|}{v_i^{-,\varepsilon}}
\end{aligned}$$

$$+\frac{\varepsilon}{\alpha_p}\sum_{i=1}^{N}\left[|I_i|\frac{\max_{x\in I_i}|u'(x)|}{u_i^{-,\varepsilon}}\right]$$

$$\leqslant \frac{\varepsilon}{\alpha_p}\sum_{i=1}^{N}\left[1+\sum_{i=1}^{N}\left[|I_i|\frac{\max_{x\in I_i}|u'(x)|}{u_i^{-,\varepsilon}}\right]\right]\left[|I_i|\frac{\max_{x\in I_i}|v'(x)|}{v_i^{-,\varepsilon}}\right]$$

$$+\frac{\varepsilon}{\alpha_p}\sum_{i=1}^{N}\left[|I_i|\frac{\max_{x\in I_i}|u'(x)|}{u_i^{-,\varepsilon}}\right]$$

$$=\frac{\varepsilon}{\alpha_p}\sum_{i=1}^{N}\left[|I_i|\frac{\max_{x\in I_i}|u'(x)|}{u_i^{-,\varepsilon}}\right]+\frac{\varepsilon}{\alpha_p}\sum_{i=1}^{N}\left[|I_i|\frac{\max_{x\in I_i}|v'(x)|}{v_i^{-,\varepsilon}}\right]$$

$$+\frac{\varepsilon}{\alpha_p}\left(\sum_{i=1}^{N}\left[|I_i|\frac{\max_{x\in I_i}|v'(x)|}{v_i^{-,\varepsilon}}\right]\right)\left(\sum_{i=1}^{N}\left[|I_i|\frac{\max_{x\in I_i}|u'(x)|}{u_i^{-,\varepsilon}}\right]\right)$$

and so

$$\limsup_{\varepsilon\to 0_+}\alpha_p\frac{K(\varepsilon)}{\varepsilon}\leqslant\int_I\left|\frac{u'}{u}\right|+\int_I\left|\frac{v'}{v}\right|+\int_I\left|\frac{u'}{u}\right|\int_I\left|\frac{v'}{v}\right|. \tag{7}$$

From (6) we have: $\sum_{i=1}^{N}\alpha_p u_i^{-,\varepsilon}v_i^{-,\varepsilon}|I_i|\leqslant\varepsilon N$ and $\sum_{i=1}^{N}\alpha_p u_i^{+,\varepsilon}v_i^{+,\varepsilon}|I_i|\geqslant\varepsilon(N-1)$ and then

$$\sum_{i=1}^{N}\alpha_p u_i^{-,\varepsilon}v_i^{-,\varepsilon}|I_i|-\alpha_p\int_I uv\leqslant\varepsilon N-\alpha_p\int_I uv\leqslant\sum_{i=1}^{N}\alpha_p u_i^{+,\varepsilon}v_i^{+,\varepsilon}|I_i|+\varepsilon-\alpha_p\int_I uv,$$

which gives us $-K(\varepsilon)\leqslant\varepsilon N-\alpha_p\int_I uv\,dx\leqslant K(\varepsilon)+\varepsilon$ and

$$-NK(\varepsilon)\leqslant N\left(\varepsilon N-\alpha_p\int_I uv\,dx\right)\leqslant NK(\varepsilon)+\varepsilon N.$$

Using $\lim_{\varepsilon\to 0_+}(\varepsilon N(\varepsilon))=\alpha_p\int_I uv\,dx$ and (7) we obtain

$$\limsup_{\varepsilon\to 0_+}\left|N\left(\varepsilon N-\alpha_p\int_I uv\,dx\right)\right|$$
$$\leqslant\int_I uv\left(\int_I\left|\frac{u'}{u}\right|+\int_I\left|\frac{v'}{v}\right|+\int_I\left|\frac{u'}{u}\right|\int_I\left|\frac{v'}{v}\right|+\alpha_p\right). \qquad \square$$

Lemma 5 and (2) with techniques used in [1] give us the following theorem:

Theorem 6. *Let* $-\infty\leqslant a<b\leqslant\infty$ *and* $I=(a,b)$. *Let* $u\in L^{p'}(I)$, $v\in L^p(I)$ *and* $(v'/v),(u'/u)\in L^1(I)\cap C[a,b]$; *then*

$$\limsup_{n\to\infty}\left|n\left[n\rho_n(T)-\alpha_p\int_I u(x)v(x)\,dx\right]\right|$$

$$\leqslant \int_I u(x)v(x)\,dx \left[\int_I \left|\frac{v'(x)}{v(x)}\right| dx + \int_I \left|\frac{u'(x)}{u(x)}\right| dx + 2\alpha_p \right.$$

$$\left. + \int_I \left|\frac{u'(x)}{u(x)}\right| dx \int_I \left|\frac{v'(x)}{v(x)}\right| dx \right],$$

where $\rho_n(T)$ stands for any of the following: $a_n(T)$, $d_n(T)$, $d^n(T)$ or $b_n(T)$, and T is the Hardy-type operator.

From Theorem 6 we have the following information about the second asymptotic:

$$\rho_n(T) = \frac{1}{n}\alpha_p \int_I u(x)v(x)\,dx + O(n^{-2}).$$

Remark 7. We have found that our method which is based on studying of behavior $\varepsilon N(\varepsilon)$ cannot be improved beyond the second term.

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